Verovatnoće, očekivane vrednosti, relacije neodređen

Neka je sistem u stanju $|\psi\rangle$

Neka se meri opservabla $\hat{A}$ sa spektralnom formom

$$\hat{A} = \sum_n a_n |\varphi_n\rangle\langle\varphi_n| = \sum_n a_n \hat{P}_n$$

koja ima nedegenerisan spektar. Verovatnoća da bude izmerena svojstvena vrednost a_n je

$$W(\hat{A}, a_n, |\psi\rangle) = \langle\psi|\hat{P}_n|\psi\rangle$$

Verovatnoća se računa isto i ako je degenerisan spektar, samo što je $\hat{P}_n = \sum_{i=1}^{g_n} |\varphi_i\rangle\langle\varphi_i|$

Ako je u pitanju opservabla sa kontinualnim spektrom, verovatnoća da se dobije rezultat iz intervala $[b,c]$ je

$$W(\hat{A}, a\in[b,c], |\psi\rangle) = \langle\psi|\hat{P}_{[b,c]}|\psi\rangle$$

gde $\hat{P}_{[b,c]} = \int_b^c |a\rangle da \langle a|$ se naziva spektralnom merom.

- $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ - očekivana vrednost

- Disperzija $\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$

- Svake dve opservable koje međusobno ne komutiraju zadovoljavaju relaciju neodređenosti

$$\Delta\hat{A}\cdot\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$$

1. Za dato stanje $|\psi\rangle$ u eksplicitnom zapisu dati verovatnoću merenja a_n iz diskretnog spektra opservable $\hat{A}$, u:

a) koordinatnoj

b) impulsnoj

c) proizvoljnoj diskretnoj reprezentaciji (koja nije A-reprezentacija). Zasebno razmatrati slučaj:

1) nedegenerisane

2) degenerisane

svojstvene vrednosti.

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1)

a) $$W(\hat{A}, a_n, |\psi\rangle) = |\langle a_n|\psi\rangle|^2$$

$$\langle a_n|\psi\rangle = \langle a_n|\hat{I}|\psi\rangle = \langle a_n|\int_{-\infty}^{+\infty}|x\rangle dx\langle x|\psi\rangle =$$

$$= \int_{-\infty}^{+\infty} a_n^*(x)\psi(x)\,dx$$

$$W(\hat{A}, a_n, |\psi\rangle) = \left|\int_{-\infty}^{+\infty} a_n^*(x)\psi(x)\,dx\right|^2$$

b) $$W(\hat{A}, a_n|\psi\rangle) = \left|\int_{-\infty}^{+\infty} a_n^*(p_x)\psi(p_x)\,dp_x\right|^2$$

c) $|\chi_e\rangle$ – druga diskretna reprezentacija

$\hookrightarrow |a_n\rangle = \sum_e d_{ne}|\chi_e\rangle$, $|\psi\rangle = \sum_e c_e|\chi_e\rangle$

$$\langle a_n|\psi\rangle = \langle a_n|\hat{I}|\psi\rangle = \langle a_n|\sum_e|\chi_e\rangle\langle\chi_e|\psi\rangle$$

$$= \sum_e d_{ne}^* c_e \Rightarrow W = \left|\sum_e d_{ne}^* c_e\right|^2$$

$$\hat{P}_n = \sum_{i=1}^{g_n} |a_i\rangle\langle a_i|$$

a) $W(\hat{A}, a_n, |\psi\rangle) = \langle\psi|\hat{P}_n|\psi\rangle = \sum_{i=1}^{g_n} |\langle a_i|\psi\rangle|^2$

$$\langle a_i|\psi\rangle = \langle a_i|\hat{I}|\psi\rangle = \langle a_i|\int_{-\infty}^{+\infty} |x\rangle dx \langle x|\psi\rangle =$$

$$= \int_{-\infty}^{+\infty} a_i^*(x)\psi(x)dx$$

$$W(\hat{A}, a_n, |\psi\rangle) = \sum_{i=1}^{g_n} \left| \int_{-\infty}^{+\infty} a_i^*(x)\psi(x)\, dx \right|^2$$

b) Domaći

c) Domaći

2. Isto kao u prethodnom zadatku, samo za svojstvenu vrednost od $\hat{A}$ iz kontinualnog spektra, uzimajući da $a \in [\alpha, \beta]$. Šta se može reći o verovatnoći merenja oštre vrednosti iz neprekidnog spektra?

Opservabla kontinualna sa diskretnom degeneracijom

$$\hat{A}|a\rangle = a|a\rangle \qquad \hat{P}_{[\alpha,\beta]} = \int_{\alpha}^{\beta} |a\rangle\langle a|\, da$$

$$W(\hat{A}, |\psi\rangle, a \in [\alpha,\beta]) = \langle\psi|\hat{P}_{[\alpha,\beta]}|\psi\rangle$$

a) Kontinualna reprezentacija $(\hat{x})$

$$W(\ldots) = \int_{\alpha}^{\beta} |\langle a|\psi\rangle|^2\, da$$

$$\langle a|\psi\rangle = \langle a|\hat{I}|\psi\rangle = \int_{-\infty}^{+\infty} \langle a|x\rangle\langle x|\psi\rangle\, dx$$

$$= \int_{-\infty}^{+\infty} a^*(x)\, \psi(x)\, dx$$

$$W(\ldots) = \int_{\alpha}^{\beta} \left| \int_{-\infty}^{+\infty} a^*(x)\, \psi(x)\, dx \right|^2 da$$

b) Kontinualna reprezentacija $\hat{P}_x$

$$W(\ldots) = \int_{\alpha}^{\beta} \left| \int_{-\infty}^{+\infty} a^*(P_x)\, \psi(P_x)\, dP_x \right|^2 da$$

c) Diskretna reprezentacija (nedegenerisana opservabla)

$$W(\ldots) = \int_\alpha^\beta |\langle a|\Psi\rangle|^2 \, da$$

$$\langle a|\Psi\rangle = \langle a|\hat{I}|\Psi\rangle = \langle a|\sum_m |\phi_m\rangle\langle\phi_m|\Psi\rangle$$

$$= \sum_m \langle a|\phi_m\rangle \langle\phi_m|\Psi\rangle$$

$$= \sum_m \phi_m(a)\, c_m$$

$$W(\ldots) = \int_\alpha^\beta \Big|\sum_m \phi_m(a) c_m\Big|^2 da$$

Merenje oštre vrednosti

$$\hat{P}_{[\alpha,\beta]} = \hat{P}_{[\alpha,\alpha]} = \int_\alpha^\alpha |a\rangle\langle a|\, da = 0 \Rightarrow W = 0$$

Kontinualna opservabla sa diskretnom degeneracijom

Primer: Slobodna čestica koja ima kontinualan spektar ($\hat{P}_x^2$) ali može ići "levo ili desno" pa jednoj energiji odgovaraju dve talasne f-je (v. zadatak u I glavi)

5. Naći verovatnoću da 3D čestica koja se nalazi u stanju $\psi(x, p_y, z)$ pri merenju kompatibilnih observabli $\hat{p}_x$ i $\hat{y}$ za bilo koje $\hat{z}$, dâ rezultat $p_x \in [p_x, p_x + \Delta p_x]$ i $y \in [y, y + \Delta y]$.

Da bi se uprostilo pisanje, neka je u gornjim intervalima ovakvo označavanje:

$$p_x \in [p_1, p_2] \qquad i \qquad y \in [y_1, y_2]$$

$$W(\hat{p}_x, \hat{y}, |\psi\rangle ; p_x \in [p_1, p_2], y \in [y_1, y_2]) =$$

$$= \langle \psi | \hat{P}_v | \psi \rangle$$

$$\hat{P}_v = \hat{P}_{[p_1, p_2]} \otimes P_{[y_1, y_2]} \otimes \hat{I}_z$$

$$\hat{P}_{[p_1, p_2]} = \int_{p_1}^{p_2} |p_x\rangle\langle p_x| \, dp_x$$

$$\hat{P}_{[y_1, y_2]} = \int_{y_1}^{y_2} |y\rangle\langle y| \, dy$$

$$\hat{I}_z = \int_{-\infty}^{+\infty} |z\rangle\langle z| \, dz$$

$$W(\ldots) = \langle \psi | \hat{P}_v | \psi \rangle = \langle \psi | \int_{p_1}^{p_2} |p_x\rangle\langle p_x| dp_x \otimes \int_{y_1}^{y_2} |y\rangle\langle y| dy \otimes$$

$$\otimes \int_{-\infty}^{+\infty} |z\rangle\langle z| dz \, |\psi\rangle$$

$$= \int_{p_1}\int_{y_1}\int_{-\infty} \Psi^*(p_x, y, z)\,\Psi(p_x, y, z)\, dp_x\, dy\, dz$$

$$= \int_{p_1}^{p_2}\int_{y_1}^{y_2}\int_{-\infty}^{+\infty} |\Psi(p_x, y, z)|^2\, dp_x\, dy\, dz$$

Ali, 3D čestica se nalazi u stanju $\Psi(x, p_y, z)$. Dakle, $\Psi(p_x, y, z)$ treba izraziti preko te talasne f-je.

$$\Psi(p_x, y, z) = \langle p_x|\langle y|\langle z|\Psi\rangle =$$

$$= \langle p_x|\langle y|\langle z|\ \hat{I}_x \otimes \hat{I}_{p_y} \otimes \hat{I}_z\ |\Psi\rangle =$$

$$\langle p_x|\langle y|\langle z| \int_{-\infty}^{+\infty} |x\rangle\langle x|\, dx \otimes \int_{-\infty}^{+\infty} |p_y\rangle\langle p_y|\, dp_y \int_{-\infty}^{+\infty} |z'\rangle\langle z'|\, dz' |\Psi\rangle$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle p_x|x\rangle\ \langle y|p_y\rangle\ \langle z|z'\rangle\ \Psi(x, p_y, z')\ dx\, dp_y\, dz'$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} x p_x} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} y p_y}\ \delta(z-z')\ \Psi(x, p_y, z')\ dx\, dp_y\, dz'$$

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(y p_y - x p_x)}\ \Psi(x, p_y, z)\ dx\, dp_y$$

Dakle

$$\Psi(p_x, y, z) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(y p_y - x p_x)}\ \Psi(x, p_y, z)\ dx\, dp_y$$

Verovatnoća

$$W(\dots) = \int_{p_1}^{p_2}\int_{y_1}^{y_2}\int_{-\infty}^{+\infty} \left| \frac{1}{2\pi\hbar} \iint_{-\infty}^{+\infty} e^{\frac{i}{\hbar}(y p_y - x p_x)}\ \Psi(x, p_y, z)\, dx\, dp_y \right|^2 dp_x\, dy\, dz$$

4. Za datu opservablu $\hat{A}$ naći očekivanu vrednost u stanju $|\psi\rangle$ izraženu u:

a) koordinatnoj

b) impulsnoj

c) nekoj diskretnoj reprezentaciji (koja nije A - reprezentacija).

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

a) $$\langle \psi | \hat{I} \hat{A} \hat{I} | \psi \rangle = \langle \psi | \int_{-\infty}^{+\infty} |x\rangle dx \langle x | \hat{A} \int_{-\infty}^{+\infty} |x'\rangle dx' \langle x' | \psi \rangle$$

$$= \iint_{-\infty}^{+\infty} \psi^*(x)\, A(x,x')\, \psi(x')\, dx\, dx'$$

b) $$\langle \psi | \hat{I} \hat{A} \hat{I} | \psi \rangle = \langle \psi | \int_{-\infty}^{+\infty} |p_x\rangle dp_x \langle p_x | \hat{A} \int_{-\infty}^{+\infty} |p_x'\rangle dp_x' \langle p_x' | \psi \rangle$$

$$= \iint_{-\infty}^{+\infty} \psi^*(p_x)\, A(p_x, p_x')\, \psi(p_x')\, dp_x\, dp_x'$$

c) $$\langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{I} \hat{A} \hat{I} | \psi \rangle = \langle \psi | \sum_m |m\rangle\langle m | \hat{A} \cdot$$

$$\sum_{m'} |m'\rangle\langle m' | \psi \rangle = \sum c_m^* A_{mm'} c_{m'}$$

$$= \sum_m c_m |m\rangle \quad , \quad \hat{A}|m\rangle \neq a_m |m\rangle$$

5. Naći disperziju operatora $\hat{A}$, zadatog matricom $\hat{A} - A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ u stanju $|\psi\rangle$ koje je u istoj reprezentaciji zadato matričnom kolonom $|\psi\rangle \to \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$.

$$\Delta\hat{A} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$$

$$= \sqrt{\langle\psi|\hat{A}^2|\psi\rangle - (\langle\psi|\hat{A}|\psi\rangle)^2}$$

$$\hat{A}^2 \to A^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\langle\psi|\hat{A}|\psi\rangle \to \frac{1}{\sqrt{2}} (-i, 1) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} =$$

$$= \frac{1}{2} (-i, 1) \begin{pmatrix} -i \\ -1 \end{pmatrix} = \frac{1}{2} (-1 - 1) = -1$$

Iskorišćeno je $\langle\psi| \to \psi^+ = \frac{1}{\sqrt{2}} (-i, 1)$

$$\langle\psi|\hat{A}^2|\psi\rangle = \langle\psi|\hat{I}|\psi\rangle = \langle\psi|\psi\rangle = \frac{1}{2} (-i, 1) \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= 1$$

$\Delta\hat{A} = 0 \Rightarrow$ stanje $|\psi\rangle$ je svojstveno za $\hat{A}$

Domaći: Rešiti svojstveni problem za $\hat{A}$

6. Naći disperziju opservable $\hat{r}$ u stanju

a) $\psi(r) = C_1\, e^{-Zr/a_0}$

b) $\psi(r) = c_2 \left(1 - \frac{Zr}{2a_0}\right) e^{-Zr/a_0}$ (za pismeni)

Prethodno normirati stanja u skladu sa pravilom normiranja

$$\langle\psi|\psi\rangle = \int_0^{+\infty} |\psi(r)|^2 r^2 dr = 1$$

$$\Delta\hat{r} = \sqrt{\langle\hat{r}^2\rangle - \langle\hat{r}\rangle^2}$$

$$\langle\hat{r}\rangle = \langle\psi|\hat{r}|\psi\rangle$$

$$|\psi\rangle \to \psi(r), \quad \hat{r} \to r, \quad \hat{I}_r = \int_0^{+\infty} |r\rangle\langle r|\, r^2 dr$$

$$\langle\hat{r}\rangle = \langle\psi|\hat{I}\hat{r}|\psi\rangle = \langle\psi|\left(\int_0^{+\infty} |r\rangle\langle r|\, r^2 dr\right)\hat{r}|\psi\rangle =$$

$$= \int_0^{+\infty} \psi^*(r)\, \langle r|\hat{r}|\psi\rangle\, r^2 dr$$

$$= \int_0^{+\infty} \psi^*(r)\, r\, \langle r|\psi\rangle\, r^2 dr$$

$$= \int_0^{+\infty} \psi^*(r)\, r\, \psi(r)\, r^2 dr$$

Slično kao gore

$$\langle\hat{r}^2\rangle = \int_0^{+\infty} \psi^*(r)\, r^2 \psi(r)\, r^2 dr$$

Normirange

$$\langle\psi|\psi\rangle = \int_0^{+\infty} |\psi(r)|^2 r^2 dr = \int_0^{+\infty} |C_1|^2 \left|e^{-\frac{Zr}{a_0}}\right|^2 r^2 dr =$$

$$= |C_1|^2 \int_0^{+\infty} e^{-\frac{2Zr}{a_0}} r^2 dr$$

Smena $\frac{2Zr}{a_0} = t \Rightarrow r = \frac{t a_0}{2Z} \Rightarrow r^2 = \frac{t^2 a_0^2}{4Z^2}$

$dr = \frac{a_0}{2Z} dt$

$$\langle\psi|\psi\rangle = |C_1|^2 \left(\frac{a_0}{2Z}\right)^3 \int_0^{+\infty} e^{-t} t^2 dt = |C_1|^2 \left(\frac{a_0}{2Z}\right)^3 \Gamma(3)$$

$$= |C_1|^2 \left(\frac{a_0}{2Z}\right)^3 2 = 1 \Rightarrow$$

$$|C_1| = \left(\frac{1}{2}\left(\frac{2Z}{a_0}\right)^3\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \left(\frac{2Z}{a_0}\right)^{\frac{3}{2}}$$

$$\langle\hat{r}\rangle = \int_0^{+\infty} r|\psi(r)|^2 r^2 dr = \int_0^{+\infty} |C_1|^2 e^{-\frac{2Zr}{a_0}} r^3 dr$$

$$= |C_1|^2 \int_0^{+\infty} e^{-\frac{2Zr}{a_0}} r^3 dr$$

Smena $\frac{2Zr}{a_0} = t \Rightarrow r = t\frac{a_0}{2Z} \Rightarrow r^3 = t^3\left(\frac{a_0}{2Z}\right)^3$

$\Rightarrow dr = \frac{a_0}{2Z} dt$

$$\langle\hat{r}\rangle = |C_1|^2 \int_0^{+\infty} e^{-t} t^3 \left(\frac{a_0}{2Z}\right)^3 \frac{a_0}{2Z} dt$$

$$= |C_1|^2 \left(\frac{a_0}{2Z}\right)^4 \int_0^{+\infty} e^{-t} t^3 dt = \frac{1}{2}\left(\frac{2Z}{a_0}\right)^3 \left(\frac{a_0}{2Z}\right)^4 \Gamma(4)$$

$$\langle \hat{r} \rangle = \frac{1}{2}\,\frac{a_0}{2Z}\cdot 6 = \frac{3a_0}{2Z}$$

$$\langle \hat{r}^2 \rangle = \int_0^{+\infty} r^4 \,|\psi(r)|^2 dr = |C_1|^2 \int_0^{+\infty} r^4 e^{-\frac{2Zr}{a_0}} dr =$$

$$= \frac{1}{2}\left(\frac{2Z}{a_0}\right)^3 \int_0^{+\infty} r^4 e^{-\frac{2Zr}{a_0}} dr$$

Smena

$$\frac{2Zr}{a_0} = t \quad \longrightarrow \quad r^4 = t^4 \left(\frac{a_0}{2Z}\right)^4, \qquad dr = \frac{a_0}{2Z}\, dt$$

$$\langle \hat{r}^2 \rangle = \frac{1}{2}\left(\frac{2Z}{a_0}\right)^3 \int_0^{+\infty} t^4 \left(\frac{a_0}{2Z}\right)^4 e^{-t} \frac{a_0}{2Z}\, dt$$

$$= \frac{1}{2}\left(\frac{2Z}{a_0}\right)^3 \left(\frac{a_0}{2Z}\right)^5 \int_0^{+\infty} e^{-t} t^4\, dt$$

$$= \frac{1}{2}\left(\frac{a_0}{2Z}\right)^2 \Gamma(5) = \frac{1}{2}\left(\frac{a_0}{2Z}\right)^2 4! = 12\left(\frac{a_0}{2Z}\right)^2$$

$$\Delta \hat{r} = \sqrt{12\left(\frac{a_0}{2Z}\right)^2 - \frac{9}{\cancel{4}}\left(\frac{a_0}{2Z}\right)^2} = \sqrt{3\left(\frac{a_0}{2Z}\right)^2}$$

$$= \sqrt{3}\,\frac{a_0}{2Z}$$

I/ Naći proizvod disperzija opservabli $\hat{x}$ i $\hat{p}_x$ za 1D česticu u stanju $\psi(x) = C e^{-\frac{x^2}{2(\Delta x)^2}}$

$$\Delta\hat{x}\cdot\Delta\hat{p}_x = ?$$

$$\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} \qquad \Delta\hat{p}_x = \sqrt{\langle\hat{p}_x^2\rangle - \langle\hat{p}_x\rangle^2}$$

$$\langle\hat{x}\rangle = \int_{-\infty}^{+\infty} x\,|\psi(x)|^2 dx = |C|^2\int_{-\infty}^{+\infty} x\, e^{-\frac{x^2}{(\Delta x)^2}} dx = 0$$

$$\langle\hat{x}^2\rangle = \int_{-\infty}^{+\infty} x^2\,|\psi(x)|^2 dx = |C|^2\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{(\Delta x)^2}} dx =$$

$$= 2|C|^2\int_0^{+\infty} x^2 e^{-\frac{x^2}{(\Delta x)^2}} dx =$$

$$\Big\lceil \text{Smena} \quad \left(\frac{x}{\Delta x}\right)^2 = t \Rightarrow \frac{x}{\Delta x} = \sqrt{t} \Rightarrow dx = \cancel{2}\Delta x\,\frac{dt}{2\sqrt{t}}$$

$$= 2|C|^2\int_0^{+\infty}(\Delta x)^2 t\, e^{-t}\,\Delta x\,\frac{dt}{2\sqrt{t}}$$

$$= \cancel{2}|C|^2(\Delta x)^3\int_0^{+\infty} t^{1/2} e^{-t} dt = \cancel{2}|C|^2(\Delta x)^3\,\Gamma\left(\frac{3}{2}\right)$$

$$= \cancel{2}|C|^2(\Delta x)^3\,\frac{1}{2}\sqrt{\pi} = \cancel{|C|^2(\Delta x)^3\sqrt{\pi}}$$

$$\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \sqrt{|C|^2(\Delta x)^3\pi^{1/2}} = \frac{|C|(\Delta x)^{\frac{3}{2}}\pi^{\frac{1}{4}}}{2^{\frac{1}{2}}}$$

$$\langle \hat{p}_x \rangle = \langle \psi | \hat{p}_x | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x)\, dx$$

$$= |C|^2 (-i\hbar) \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(\Delta x)^2}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2(\Delta x)^2}} dx =$$

$$= |C|^2 (-i\hbar) \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(\Delta x)^2}} \left(-\frac{2x}{2(\Delta x)^2}\right) e^{-\frac{x^2}{2(\Delta x)^2}} dx =$$

$$= \frac{i\hbar |C|^2}{(\Delta x)^2} \int_{-\infty}^{+\infty} x\, e^{-\frac{x^2}{(\Delta x)^2}} dx = 0$$

$$\langle \hat{p}_x^2 \rangle = -\hbar^2 |C|^2 \int_{-\infty}^{+\infty} \psi^*(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx =$$

$$\cancel{= -\hbar^2 |C|^2 \int_{-\infty}^{+\infty} \psi^*(x) \frac{\partial^2}{\partial x^2}}$$

$$= -\hbar^2 |C|^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(\Delta x)^2}} \frac{\partial^2 e^{-\frac{x^2}{2(\Delta x)^2}}}{\partial x^2} dx =$$

$$= -\hbar^2 |C|^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(\Delta x)^2}} \frac{\partial}{\partial x} \left(-\frac{x}{(\Delta x)^2} e^{-\frac{x^2}{2(\Delta x)^2}}\right) dx =$$

$$= +\frac{\hbar^2 |C|^2}{(\Delta x)^2} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(\Delta x)^2}} \left(e^{-\frac{x^2}{2(\Delta x)^2}} + x\left(-\frac{x}{(\Delta x)^2} e^{-\frac{x^2}{2(\Delta x)^2}}\right)\right) dx$$

$$= \frac{|C|^2 \hbar^2}{(\Delta x)^2} \left[\int_{-\infty}^{+\infty} e^{-\frac{x^2}{(\Delta x)^2}} dx - \frac{1}{(\Delta x)^2} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{(\Delta x)^2}} dx \right]$$

$$= \frac{|C|^2 \hbar^2}{(\Delta x)^2} \left[\Delta x \sqrt{\pi} - \frac{2}{(\Delta x)^2} \int_{0}^{+\infty} x^2 e^{-\frac{x^2}{(\Delta x)^2}} dx \right]$$

Već računato!

$$\frac{|C|^2 \hbar^2}{(\Delta x)^2} \left[\Delta x \sqrt{\pi} - \frac{1}{(\Delta x)^2} (\Delta x)^3 \frac{\sqrt{\pi}}{2} \right]$$

$$\langle p_x^2 \rangle = \frac{|C|^2 \hbar^2}{(\Delta x)^2} \, \frac{\Delta x \sqrt{\pi}}{2} = \frac{|C|^2 \hbar^2}{\Delta x} \, \frac{\sqrt{\pi}}{2}$$

$$\Delta \hat{p}_x = \sqrt{\langle \hat{p}_x^2 \rangle - \langle \hat{p}_x \rangle^2} = \sqrt{\langle \hat{p}_x^2 \rangle} =$$

$$= \frac{|C| \hbar}{(\Delta x)^{\frac{1}{2}}} \, \frac{\pi^{\frac{1}{4}}}{2^{\frac{1}{2}}}$$

Normiranje

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

$$\boxed{\psi(x) = \frac{1}{\sqrt{\Delta x \sqrt{\pi}}} \, e^{-\frac{x^2}{2(\Delta x)^2}}}$$

$$|C|^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{(\Delta x)^2}} dx = 1$$

$$|C|^2 \left(\Delta x \sqrt{\pi} \right) = 1 \quad \rightarrow$$

$$|C| = \frac{1}{(\Delta x)^{\frac{1}{2}} \, \pi^{1/4}}$$

$$\boxed{\Delta \hat{x} \, \Delta \hat{p}_x} = |C| \, (\Delta x)^{\frac{3}{2}} \, \frac{\pi^{\frac{1}{4}}}{2^{1/2}} \, \frac{|C| \hbar}{(\Delta x)^{\frac{1}{2}}} \, \frac{\pi^{\frac{1}{4}}}{2^{1/2}}$$

$$= |C|^2 \, \Delta x \, \frac{\pi^{1/2}}{2} \, \hbar$$

$$= \frac{1}{\cancel{\Delta x \, \pi^{1/2}}} \, \cancel{\Delta x} \, \cancel{\pi^{1/2}} \, \frac{\hbar}{2} = \boxed{\frac{\hbar}{2}}$$

8. Dato je stanje $\psi(x) = c\, e^{-\frac{\pi \varkappa (x-b)^2}{\hbar} + \frac{2\pi i \rho}{\hbar}}$
Normirati dato stanje u prostoru $L_2(-\infty, +\infty)$, a zatim dokazati da je proizvod disperzija $\Delta\hat{x}\, \Delta\hat{p}_x = \frac{\hbar}{2}$. $\rho, b, \varkappa \in \mathbb{R}$

Prostor $L_2(-\infty,+\infty) = \left\{ \psi(x) \,\middle|\, \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \right\}$
$\lim_{x\to\pm\infty} \psi(x) = 0 \qquad (x \in \mathbb{R})$

Ako je $\psi(x) = c\; e^{-\pi\varkappa(x-b)^2/\hbar + 2\pi i\rho/\hbar}$

onda je $|\psi(x)|^2 = |c|^2\, e^{-2\pi\varkappa(x-b)^2/\hbar}$ a

$|e^{2\pi i\rho/\hbar}| = 1$

Normiranje

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = |c|^2 \int_{-\infty}^{+\infty} e^{-2\pi\varkappa(x-b)^2/\hbar}\, dx$$

$$= |c|^2 \int_{-\infty}^{+\infty} e^{-\alpha(x-b)^2} dx \quad ; \; \alpha = \frac{2\pi\varkappa}{\hbar}$$

Smena $x - b = t \qquad dx = dt$

$$= |c|^2 \int_{-\infty}^{+\infty} e^{-\alpha t^2} dt$$

$$= |c|^2 \sqrt{\frac{\pi}{\alpha}} = 1$$

$$|c| = \left(\frac{\alpha}{\pi}\right)^{\frac14} = \left(\frac{2\pi\varkappa}{\pi\hbar}\right)^{\frac14} = \left(\frac{2\varkappa}{\hbar}\right)^{\frac14}$$

$$C = |C|\, e^{i\delta} = \left(\frac{2\mu}{\hbar}\right)^{\frac{1}{4}} e^{i\delta} \qquad (\delta = 0)$$

$$\langle \hat{x} \rangle = \int_{-\infty}^{+\infty} x\, |\psi(x)|^2\, dx$$

$$= |C|^2 \int_{-\infty}^{+\infty} x\, e^{-2\pi\mu(x-b)^2/\hbar}\, dx = b$$

$$\langle \hat{x}^2 \rangle = |C|^2 \int_{-\infty}^{+\infty} x^2 e^{-2\pi\mu(x-b)^2/\hbar}\, dx$$

$$= |C|^2 \int_{-\infty}^{+\infty} x^2 e^{-\alpha(x-b)^2}\, dx \quad ; \quad x - b = u, \quad dx = du$$

$$= |C|^2 \int_{-\infty}^{+\infty} (b+u)^2 e^{-\alpha u^2}\, du$$

$$= |C|^2 \int_{-\infty}^{+\infty} (b^2 + 2bu + u^2)\, e^{-\alpha u^2}\, du$$

$$= |C|^2 \left[b^2 \int_{-\infty}^{+\infty} e^{-\alpha u^2} du + \cancel{2b \int_{-\infty}^{+\infty} u\, e^{-\alpha u^2} du}^{\,0} + \int_{-\infty}^{+\infty} u^2 e^{-\alpha u^2} du \right]$$

$$= |C|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \int_{-\infty}^{+\infty} u^2 e^{-\alpha u^2} du \right]$$

smena
$u^2 = t$
$u = \sqrt{t}$
$du = \frac{dt}{2\sqrt{t}}$

$$= |C|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + 2 \int_{0}^{+\infty} u^2 e^{-\alpha u^2} du \right]$$

$$= |C|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + 2 \int_{0}^{+\infty} t\, e^{-\alpha t} \frac{dt}{2\sqrt{t}} \right]$$

$$= |C|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \int_{0}^{+\infty} t^{1/2} e^{-\alpha t}\, dt \right] \qquad \alpha t = \xi$$

$$= |c|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \int_0^{+\infty} \frac{\xi^{1/2}}{\alpha^{1/2}} e^{-\xi} \frac{d\xi}{\alpha} \right]$$

$$= |c|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \frac{1}{\alpha^{3/2}} \int_0^{+\infty} \xi^{1/2} e^{-\xi} d\xi \right]$$

$$= |c|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \frac{1}{\alpha^{3/2}} \Gamma\left(\frac{3}{2}\right) \right]$$

$$= |c|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \frac{\sqrt{\pi}}{2\alpha^{3/2}} \right]$$

$$= |c|^2 \left[b^2 \sqrt{\frac{\pi}{\alpha}} + \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \right]$$

$$= \underbrace{|c|^2 \sqrt{\frac{\pi}{\alpha}}}_{1} \left[b^2 + \frac{1}{2\alpha} \right]$$

$$\boxed{\Delta \hat{x} = \sqrt{b^2 + \frac{1}{2\alpha} - b^2} = \frac{1}{\sqrt{2\alpha}}}$$

$$\langle \hat{p}_x \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi(x)\, dx$$

$$= |c|^2 \int_{-\infty}^{+\infty} e^{-\pi\varkappa(x-b)^2/\hbar - 2\pi i s/\hbar} \left(-i\hbar \frac{d}{dx}\right) e^{-\pi\varkappa(x-b)^2/\hbar + 2\pi i s/\hbar}\, dx$$

$$= -i\hbar |c|^2 \int_{-\infty}^{+\infty} e^{-\pi\varkappa(x-b)^2/\hbar - 2\pi i s/\hbar} \left(\frac{-\pi\varkappa}{\hbar}\right) 2(x-b)\, e^{-\pi\varkappa(x-b)^2/\hbar + 2\pi i s/\hbar}\, dx$$

$$= 2i\varkappa\pi |c|^2 \int_{-\infty}^{+\infty} (x-b)\, e^{-2\pi\varkappa(x-b)^2/\hbar}\, dx = 0$$

$$\langle \hat{p}_x^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi(x)\, dx$$

$$\langle P_x^2\rangle = |c|^2\int_{-\infty} e^{-\pi\mu(x-b)^2/\hbar - 2\pi i\rho/\hbar}\left(-\hbar^2\frac{d^2}{dx^2}\right)e^{-\pi\mu(x-b)^2/\hbar + 2\pi i\rho/\hbar}\, dx$$

$$= -|c|^2\hbar^2\int_{-\infty}^{+\infty} e^{-\pi\mu(x-b)^2/\hbar - 2\pi i\rho/\hbar}\frac{d}{dx}\left(-\frac{\pi\mu}{\hbar}2(x-b)\, e^{-\pi\mu(x-b)^2/\hbar + 2\pi i\rho}\right.$$

$$= 2|c|^2\hbar\pi\mu\int_{-\infty}^{+\infty} e^{-\pi\mu(x-b)^2/\hbar - 2\pi i\rho/\hbar}\frac{d}{dx}\left((x-b)\,e^{-\pi\mu(x-b)^2/\hbar + 2\pi i\rho/\hbar}\right)dx$$

$$= 2|c|^2\hbar\pi\mu\int_{-\infty}^{+\infty} e^{-\pi\mu(x-b)^2/\hbar - 2\pi i\rho/\hbar}\left(e^{-\pi\mu(x-b)^2/\hbar + 2\pi i\rho/\hbar} + (x-b)\left(-\frac{\pi\mu}{\hbar}2(x-b)\, e^{-\pi\mu(x-b)^2/\hbar + 2\pi i\rho/\hbar}\right)\right)dx$$

$$2|c|^2\hbar\pi\mu\int_{-\infty}^{+\infty}\left(e^{-2\pi\mu(x-b)^2/\hbar} - 2\frac{\pi\mu}{\hbar}(x-b)^2\, e^{-2\pi\mu(x-b)^2/\hbar}\right)dx$$

$$2|c|^2\hbar\pi\mu\left[\int_{-\infty}^{+\infty} e^{-2\pi\mu(x-b)^2/\hbar}\,dx - 2\frac{\pi\mu}{\hbar}\int_{-\infty}^{+\infty}(x-b)^2 e^{-2\pi\mu(x-b)^2/\hbar}\,dx\right]$$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\mu}{\hbar}\int_{-\infty}^{+\infty}(x-b)^2 e^{-2\pi\mu(x-b)^2/\hbar}\,dx\right]$$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\mu}{\hbar}\int_{-\infty}^{+\infty}\xi^2 e^{-2\pi\mu\xi^2/\hbar}\,d\xi\right]$$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{4\pi\mu}{\hbar}\int_{0}^{+\infty}\xi^2 e^{-\alpha\xi^2}\,d\xi\right]$$

$\xi^2 = t$

$\xi = \sqrt{t}$

$d\xi = \frac{dt}{2\sqrt{t}}$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{\cancel{4}\pi\mu}{\hbar}\int_{0}^{+\infty} t\, e^{-\alpha t}\frac{dt}{\cancel{2}\sqrt{t}}\right]$$

$\alpha t = x$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\mu}{\hbar}\int_{0}^{+\infty}\left(\frac{x}{\alpha}\right)^{\frac{1}{2}} e^{-x}\frac{dx}{\alpha}\right]$$

$$2|c|^2\hbar\pi\mu\left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\mu}{\hbar\alpha^{3/2}}\int_{0}^{+\infty} x^{\frac{1}{2}} e^{-x}\,dx\right]$$

$$= 2|c|^2 \hbar \pi \varkappa \left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\varkappa}{\hbar \alpha^{3/2}} \Gamma\left(\frac{3}{2}\right) \right]$$

$$= 2|c|^2 \hbar \pi \varkappa \left[\sqrt{\frac{\pi}{\alpha}} - \frac{2\pi\varkappa}{\hbar \alpha^{3/2}} \frac{1}{2}\sqrt{\pi} \right]$$

$$= 2|c|^2 \hbar \pi \varkappa \sqrt{\frac{\pi}{\alpha}} \left[1 - \frac{\pi\varkappa}{\hbar\alpha} \right]$$

$$\boxed{\alpha = \frac{2\pi\varkappa}{\hbar}}$$

$$= 2 \underbrace{|c|^2 \sqrt{\frac{\pi}{\alpha}}}_{1} \hbar\pi\varkappa \left[1 - \frac{\cancel{\pi}\varkappa}{\cancel{\hbar}\frac{2\cancel{\pi}\varkappa}{\cancel{\hbar}}} \right]$$

$$= 2\hbar\pi\varkappa \frac{1}{2} = \hbar\pi\varkappa$$

odnosno izraženo preko α

$$\hbar\pi\varkappa = \frac{\alpha\hbar^2}{2}$$

$$\Delta \hat{p}_x = \sqrt{\frac{\alpha\hbar^2}{2}}$$

$$\Delta\hat{x} \cdot \Delta\hat{p}_x = \frac{1}{\sqrt{2\alpha}} \cdot \sqrt{\frac{\alpha\hbar^2}{2}} = \frac{\hbar}{2}$$

1. Naći najverovatniju vrednost opservable $\hat{r} \equiv |\hat{\vec{r}}|$ u stanjima

a) $\psi(r) = c_1 e^{-\frac{Zr}{a_0}}$

b) $\psi(r) = c_2 \left(1 - \frac{rZ}{2a_0}\right) e^{-\frac{Zr}{2a_0}}$. a_0, Z konstante

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$$\int_0^{+\infty} \underbrace{|\psi(r)|^2 r^2}_{g_v(r)} dr = 1$$

$$\frac{d g_v(r)}{dr} = 0 \qquad \text{ekstremum}$$

a) $g_v(r) = |c_1|^2 e^{-\frac{2Zr}{a_0}} r^2$

$$g_v'(r) = |c_1|^2 \left(-\frac{2Z}{a_0} e^{-\frac{2Zr}{a_0}} r^2 + e^{-\frac{2Zr}{a_0}} 2r \right)$$

$$= 2(c_1)^2 e^{-\frac{2Zr}{a_0}} r \left(1 - \frac{Zr}{a_0} \right)$$

$$g_v'(r) = 0 \Rightarrow r = \frac{a_0}{Z}$$

b) Domaći

b) $\psi(r) = c_2 \left(1 - \frac{Zr}{2a_0}\right) e^{-\frac{Zr}{2a_0}}$

$g_v(r) = r^2 |\psi(r)|^2$

$g_v(r) = |c_2|^2 r^2 \left(1 - \frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}}$

$$\frac{dg_v(r)}{dr} = |c_2|^2 \frac{d}{dr}\left(r^2 \underbrace{\left(1 - \frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}}}\right)$$

$$= |c_2|^2 \left(2r\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}} + r^2 \left[\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}}\right]'\right)$$

$$= |c_2|^2 \left(2r\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}} + r^2 \left[\cancel{2}\left(1-\frac{Zr}{2a_0}\right)\left(-\frac{Z}{\cancel{2}a_0}\right) e^{-\frac{Zr}{a_0}} + \left(1-\frac{Zr}{2a_0}\right)^2\left(-\frac{Z}{a_0}\right) e^{-\frac{Zr}{a_0}}\right]\right) =$$

$$= |c_2|^2 \left(2r\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}} - r^2 \frac{Z}{a_0} e^{-\frac{Zr}{a_0}} \left[\left(1-\frac{Zr}{2a_0}\right) + \left(1-\frac{Zr}{2a_0}\right)^2\right]\right) =$$

$$|c_2|^2 \left(2r\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}} - \frac{Zr^2}{a_0} e^{-\frac{Zr}{a_0}} \left(1-\frac{Zr}{2a_0}\right)\left(2-\frac{Zr}{2a_0}\right)\right) =$$

$$= |c_2|^2 r \left(1-\frac{Zr}{2a_0}\right) e^{-\frac{Zr}{a_0}} \left(2\left(1-\frac{Zr}{2a_0}\right) - \frac{rZ}{a_0}\left(2-\frac{Zr}{2a_0}\right)\right)$$

$$= |c_2|^2 r \left(1-\frac{Zr}{2a_0}\right)\left(2 - \frac{Zr}{a_0} - \frac{2rZ}{a_0} + \frac{Z^2 r^2}{2a_0^2}\right)$$

$$= |c_2|^2 r \left(1-\frac{Zr}{2a_0}\right)\left(2 - \frac{3Z}{a_0} r + \frac{Z^2}{2a_0^2} r^2\right)$$

Nule prvog izvoda

$r_1 = 0$ $\qquad\qquad r_{3,4} = (3 \pm \sqrt{5})\,\frac{a_0}{Z}$

$r_2 = \frac{2a_0}{Z}$

$\ulcorner\; a = \frac{Z^2}{2a_0^2} \qquad b = -\frac{3Z}{a_0} \qquad c = 2 \;\lrcorner$

II izvod je komplikovan za procenu, pa se procenjuje numerički

$g_v(r_1) = 0 \qquad g_v(r_2) = 0$

$g_v(r_3) = 0,191\left(\frac{a_0}{Z}\right)^{-1} \qquad g_v(r_4) = 0.0519\,\frac{Z}{a_0}$

Dakle, najverovatnija vrednost opservable je

$$r = (3+\sqrt{5})\,\frac{a_0}{Z}$$

Primedba: Da bi se dobili brojčani rezultati za $g_v(r_3)$ i $g_v(r_4)$, prethodno je potrebno normirati f-ju $\psi(r) = C_2\left(1 - \frac{Zr}{2a_0}\right) e^{-\frac{Zr}{2a_0}}$ i naći C_2. Za domaći! Rezultat: $C_2 = \frac{1}{\sqrt{2}}\frac{Z^{3/2}}{a_0^{3/2}}$

2. Diskutovati relacije neodređenosti opservabli $\hat{\phi}$ i $\hat{L}_z$, koje formalno zadovoljavaju komutacionu relaciju

$$[\hat{\phi}, \hat{L}_z] = i\hbar$$

$$[\hat{\phi}, \hat{L}_z] = i\hbar \quad \overset{?}{\Rightarrow} \quad \Delta\hat{\phi}\, \Delta\hat{L}_z \geq \frac{\hbar}{2}$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \qquad \text{sv. fje} \quad h(\phi) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{i m \phi}{\hbar}}, \quad m \in \mathbb{Z}$$

v. pozadi

$$\Delta\hat{\phi} = \sqrt{\langle \hat{\phi}^2 \rangle - \langle \hat{\phi} \rangle^2}$$

$$\langle \hat{\phi}^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi^2 \, d\phi = \dots = \frac{(2\pi)^2}{3}$$

v. pozadi

$$\langle \hat{\phi} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi \, d\phi = \dots = \pi$$

$$\left.\begin{aligned} \Delta\hat{\phi} &= \dots = \frac{\pi}{\sqrt{3}} \\ \Delta\hat{L}_z &= 0 \end{aligned}\right\} \quad \Delta\hat{\phi}\, \Delta\hat{L}_z = 0 \neq \frac{\hbar}{2}$$

$$D(\hat{L}_z) = \{ h(\phi) \mid h(0) = h(2\pi),\ \hat{L}_z^{+} = \hat{L}_z \}$$

$$D(\hat{L}_z) \neq D(\hat{\phi}\hat{L}_z) = \{ \phi h(\phi) \mid \hat{L}_z^{+} \neq \hat{L}_z \}$$

U kvantnoj mehanici se $\hat{\phi}$ i $\hat{L}_z$ ne mogu smatrati kanonski konjugovanim opservablama!